

# Rate of convergence of certain families of Jain operators of integral type

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## Abstract

In the present paper, the authors introduce and investigate new sequences of positive linear operators which include some well known operators as special cases. Here we estimate the rate of convergence for functions having derivatives of bounded variation by families of Jain operators of integral type.

**Keywords:** Positive linear operators; Jain operators; Bounded variation; Rate of convergence

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## 1. Introduction

In the year 1972, Jain [1] introduced and studied the new class of positive linear operators using Poisson-type distribution as

$$G_n^\mu(f, x) = \sum_{k=0}^{\infty} nx(nx + k\mu)^{k-1} \frac{e^{-(nx+k\mu)}}{k!} f\left(\frac{k}{n}\right), \quad (1.1)$$

where  $\mu \in [0, 1)$  and  $f \in C(\mathbf{R}^+)$ . In the particular case  $\mu = 0$ ,  $G_n^0$ ,  $n \in \mathbf{N}$ , turn out to be well-known Szász-Mirakjan operators [2]. Umar and Razi [3] studied Kantorovich-type extensions of  $G_n^\mu$ . Tarabie [4] and Mishra and Patel [5] introduced integral versions of the Jain operators using Beta basis functions and discussed their approximation properties. Recently, both the authors have established the Jain-Baskakov operators and different generalizations of them in [6].

The general integral modification of Jain-Baskakov operators to approximate Lebesgue integrable functions on the interval  $[0, \infty)$ , can be defined as follows:

$$K_n^{\mu,r}(f, x) = \frac{n^r(n-r-1)!}{(n-2)!} \sum_{v=0}^{\infty} \omega_\mu(v, nx) \int_0^{\infty} p_{n-r,v+r}(t) f(t) dt, \quad r \geq 0, \quad (1.2)$$

where  $n \in \mathbf{N}$ ,  $r \in \mathbf{N} \cup \{0\}$ ,  $n > r$ ,  $\mu \in [0, 1)$  and the Baskakov and the Jain basis functions are defined as

$$p_{n,v}(t) = \binom{n+v-1}{v} \frac{t^v}{(1+t)^{n+v}},$$

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$$\omega_\mu(v, nx) = nx(nx + v\mu)^{v-1} \frac{e^{-(nx+v\mu)}}{v!}.$$

We note that, the operators  $K_n^{\mu,r}(f, \cdot)$  are linear and positive. If  $r = 0$ , the operators (1.2) are equal to the Jain-Baskakov operators studied by Patel and Mishra [6]. The rate of convergence for functions having derivatives of bounded variation is the investigated by many authors [7, 8, 9, 10, 11, 12]. Here we extend these studies to investigation of similar properties of the modified Jain-Baskakov operators as defined in (1.2).

## 2. Estimation of moments

**Lemma 1 ([1]).** For  $G_n^\mu(t^m, x)$ ,  $m = 0, 1, 2$ , we have

$$G_n^\mu(1, x) = 1, \quad G_n^\mu(t, x) = \frac{x}{1-\mu}, \quad G_n^\mu(t^2, x) = \frac{x^2}{(1-\mu)^2} + \frac{x}{n(1-\mu)^3}.$$

**Lemma 2.** Let the  $m^{\text{th}}$  order moment be defined as

$$K_{n,r,m}^\mu(x) = (n-r-1) \sum_{v=0}^{\infty} \omega_\mu(v, nx) \int_0^\infty p_{n-r,v+r}(t) t^m dt. \quad (2.1)$$

Then

$$K_{n,r,0}^\mu(x) = 1, n > r+1, \quad K_{n,r,1}^\mu(x) = \frac{nx + (r+1)(1-\mu)}{(n-r-2)(1-\mu)} \text{ for } n > r+2,$$

$$K_{n,r,2}^\mu(x) = \frac{1}{(n-r-2)(n-r-3)} \left[ \frac{n^2 x^2}{(1-\mu)^2} + \left[ \frac{n}{(1-\mu)^3} + \frac{n(2r+3)}{1-\mu} \right] x + (r+1)(r+2) \right] \text{ for } n > r+3.$$

**Proof:** For  $n > r+1$ , we have

$$\begin{aligned} K_{n,r,0}^\mu(x) &= (n-r-1) \sum_{v=0}^{\infty} \omega_\mu(v, nx) \frac{(n+v-1)!}{(v+r)!(n-r-1)!} \int_0^\infty \frac{t^{v+r}}{(1+t)^{n+v}} dt \\ &= (n-r-1) \sum_{v=0}^{\infty} \omega_\mu(v, nx) \frac{(n+v-1)!}{(v+r)!(n-r-1)!} \cdot \frac{(n-r-2)!(v+r)!}{(n+v-1)!} = G_n^\mu(1, x) = 1. \end{aligned}$$

Now, for  $m = 1$  and  $n > r+2$ , we get

$$\begin{aligned} K_{n,r,1}^\mu(x) &= (n-r-1) \sum_{v=0}^{\infty} \omega_\mu(v, nx) \frac{(n+v-1)!}{(v+r)!(n-r-1)!} \int_0^\infty \frac{t^{v+r}}{(1+t)^{n+v}} \cdot t dt \\ &= (n-r-1) \sum_{v=0}^{\infty} \omega_\mu(v, nx) \frac{(n+v-1)!}{(v+r)!(n-r-1)!} \cdot \frac{(n-r-3)!(v+r+1)!}{(n+v-1)!} \\ &= (n-r-1) \sum_{v=0}^{\infty} \omega_\mu(v, nx) \frac{v+r+1}{(n-r-1)(n-r-2)} \\ &= \frac{n}{(n-r-2)} \sum_{v=0}^{\infty} \omega_\mu(v, nx) \frac{v}{n} + \frac{r+1}{(n-r-2)} \sum_{v=0}^{\infty} \omega_\mu(v, nx) \\ &= \frac{n}{(n-r-2)} \frac{x}{(1-\mu)} + \frac{r+1}{(n-r-2)} \\ &= \frac{nx + (r+1)(1-\mu)}{(n-r-2)(1-\mu)}. \end{aligned}$$

Further, for  $m = 2$  and  $n > r + 3$ , we obtain

$$\begin{aligned}
K_{n,r,2}^\mu(x) &= (n-r-1) \sum_{v=0}^{\infty} \omega_\mu(v, nx) \frac{(n+v-1)!}{(v+r)!(n-r-1)!} \int_0^\infty \frac{t^{v+r}}{(1+t)^{n+v}} \cdot t^2 dt \\
&= (n-r-1) \sum_{v=0}^{\infty} \omega_\mu(v, nx) \frac{(n+v-1)!}{(v+r)!(n-r-1)!} \cdot \frac{(n-r-4)!(v+r+2)!}{(n+v-1)!} \\
&= (n-r-1) \sum_{v=0}^{\infty} \omega_\mu(v, nx) \frac{(v+r+1)(v+r+2)}{(n-r-1)(n-r-2)(n-r-3)} \\
&= \frac{1}{(n-r-2)(n-r-3)} \sum_{v=0}^{\infty} \omega_\mu(v, nx) (v^2 + (2r+3)v + (r+1)(r+2)) \\
&= \frac{1}{(n-r-2)(n-r-3)} [n^2 G_n^\mu(t^2, x) + n(2r+3)G_n^\mu(t, x) + (r+1)(r+2)G_n^\mu(1, x)] \\
&= \frac{1}{(n-r-2)(n-r-3)} \left[ \frac{n^2 x^2}{(1-\mu)^2} + \left[ \frac{n}{(1-\mu)^3} + \frac{n(2r+3)}{1-\mu} \right] x + (r+1)(r+2) \right].
\end{aligned}$$

This completes the proof of Lemma 2.

**Lemma 3.** Let the  $m^{\text{th}}$  order central moment be defined as

$$T_{n,r,m}^\mu(x) = (n-r-1) \sum_{v=0}^{\infty} \omega_\mu(v, nx) \int_0^\infty p_{n-r,v+r}(t) (t-x)^m dt \quad (2.2)$$

then  $T_{n,r,0}^\mu(x) = 1$  for  $n > r + 1$ ,  $T_{n,r,1}^\mu(x) = \frac{(1+r)(1-\mu) + x((2+r)(1-\mu) + n\mu)}{(n-r-2)(1-\mu)}$  for  $n > r + 2$  and

$$\begin{aligned}
T_{n,r,2}^\mu(x) &= x^2 \left( 1 + \frac{n^2}{(n-r-3)(n-r-2)(1-\mu)^2} - \frac{2n}{(n-r-2)(1-\mu)} \right) \\
&\quad + x \left( \frac{n(1+(3+2r)(1-\mu)^2)}{(n-r-3)(n-r-2)(1-\mu)^3} - \frac{2(1+r)}{n-r-2} \right) \\
&\quad + \frac{(1+r)(2+r)}{(n-r-3)(n-r-2)}, \text{ for } n > r + 3.
\end{aligned}$$

The proof of above lemma follows from linear properties of the operators  $K_n^{\mu,r}(f, \cdot)$ . Lemma 2 shows that the operators  $K_n^{\mu,r}(f, \cdot)$  does not preserve the linear functions, that is,  $K_n^{\mu,r}(f, x) \neq f(x)$  for  $f(t) = at + b$ , where  $a$  and  $b$  are real constants.

**Remark 1.** For fixed  $r$ , there is a constant  $C > 1$  (which depends only on  $r$ ) such that, for all  $n$  being sufficiently large, all  $\mu$  being sufficiently small (say  $0 < \mu < \mu_0$ ), and  $x \in (0, \infty)$ ,

$$\frac{x^2}{nC} \leq T_{n,r,2}^\mu(x) \leq \frac{Cx^2}{n}.$$

**Remark 2.** By using Cauchy-Schwarz inequality, it follows from Remark 1, that for all  $n$  being sufficiently large,

all  $\mu$  being sufficiently small,  $C > 1$  and  $x \in (0, \infty)$ , we have

$$(n-r-1) \sum_{v=0}^{\infty} \omega_{\mu}(v, nx) \int_0^{\infty} p_{n-r, v+r}(t) |t-x| dt \leq [T_{n,r,2}^{\mu}(x)]^{1/2} \leq \sqrt{\frac{Cx^2}{n}}. \quad (2.3)$$

**Lemma 4.** Let  $x \in (0, \infty)$  and  $C > 1$ , then for all  $n$  being sufficiently large and all  $\mu$  being sufficiently small, we have

$$\delta_{n,r}^{\mu}(x, y) = (n-r-1) \sum_{v=0}^{\infty} \omega_{\mu}(v, nx) \int_0^y p_{n-r, v+r}(t) dt \leq \frac{Cx^2}{n(x-y)^2}, \quad 0 \leq x < y \quad (2.4)$$

$$1 - \delta_{n,r}^{\mu}(x, z) = (n-r-1) \sum_{v=0}^{\infty} \omega_{\mu}(v, nx) \int_z^{\infty} p_{n-r, v+r}(t) dt \leq \frac{Cx^2}{n(z-x)^2}, \quad x < z < \infty. \quad (2.5)$$

**Proof:** The proof of the above lemma follows easily by using Remark 1. For instance, for the first inequality for all  $n$  being sufficiently large, all  $\mu$  being sufficiently small and  $0 \leq y < x$ , we have

$$\begin{aligned} \delta_{n,r}^{\mu}(x, y) &= (n-r-1) \sum_{v=0}^{\infty} \omega_{\mu}(v, nx) \int_0^y p_{n-r, v+r}(t) dt \\ &\leq (n-r-1) \sum_{v=0}^{\infty} \omega_{\mu}(v, nx) \int_0^y p_{n-r, v+r}(t) \frac{(t-x)^2}{(y-x)^2} dt \\ &= \frac{T_{n,r,2}^{\mu}(x)}{(y-x)^2} \leq \frac{Cx^2}{n(x-y)^2}. \end{aligned}$$

The proof of the second inequality follows along the similar lines.

### 3. Rate of Convergence

By  $DB_q(0, \infty)$  (where  $q$  is some positive integer), we mean the class of absolutely continuous functions  $f$  defined on  $(0, \infty)$  satisfying the following conditions:

- (i)  $f(t) = O(t^q)$ ,  $t \rightarrow \infty$ ;
- (ii) the function  $f$  has the first derivative on the interval  $(0, \infty)$  which coincide almost everywhere with a function which is of bounded variation on every finite subinterval of  $(0, \infty)$ . It can be observed that for all functions  $f \in DB_q(0, \infty)$ , we can have the representation

$$f(x) = f(c) + \int_c^x \psi(t) dt, \quad 0 < c \leq x.$$

**Theorem 1.** Let  $f \in DB_q(0, \infty)$ ,  $q > 0$  and  $x \in (0, \infty)$ . Then for  $C > 1$ , all  $\mu$  being sufficiently small and all  $n$

being sufficiently large, we have

$$\begin{aligned}
\left| \frac{(n-2)!}{n^r(n-r-2)!} K_n^{\mu,r}(f, x) - f(x) \right| &\leq \frac{Cx}{n} \left( \sum_{k=1}^{\lfloor \sqrt{n} \rfloor} \bigvee_{x-x/k}^{x+x/k} ((f')_x) + \frac{x}{\sqrt{n}} \bigvee_{x-x/\sqrt{n}}^{x+x/\sqrt{n}} ((f')_x) \right) \\
&+ \frac{C}{n} (|f(2x) - f(x) - xf'(x^+)| + |f(x)|) \\
&+ |f'(x^+)| \frac{Cx}{n} + \frac{1}{2} \sqrt{\frac{Cx^2}{n}} |f'(x^+) - f'(x^-)| \\
&+ \frac{1}{2} |f'(x^+) + f'(x^-)| \frac{(1+r)(1-\mu) + x((2+r)(1-\mu) + n\mu)}{(n-r-2)(1-\mu)} + O(n^{-q}),
\end{aligned}$$

where  $\bigvee_a^b f(x)$  denotes the total variation of  $f_x$  on  $[a, b]$ , and  $f_x$  is defined by

$$f_x(t) = \begin{cases} f(t) - f(x^-), & 0 \leq t < x; \\ 0, & t = x; \\ f(t) - f(x^+), & x < t < \infty. \end{cases}$$

**Proof:** Using the mean value theorem, we can write

$$\begin{aligned}
\left| \frac{(n-2)!}{n^r(n-r-2)!} K_n^{\mu,r}(f, x) - f(x) \right| &\leq (n-r-1) \sum_{v=0}^{\infty} \omega_{\mu}(v, nx) \int_0^{\infty} p_{n-r, v+r}(t) |f(t) - f(x)| dt \\
&= \int_0^{\infty} \left| \int_x^t (n-r-1) \sum_{v=0}^{\infty} \omega_{\mu}(v, nx) p_{n-r, v+r}(t) f'(u) du \right| dt.
\end{aligned}$$

Also, using the identity

$$f'(u) = \frac{f'(x^+) + f'(x^-)}{2} + (f')_x(u) + \frac{f'(x^+) - f'(x^-)}{2} \operatorname{sgn}(u - x) + \left[ f'(x) - \frac{f'(x^+) + f'(x^-)}{2} \right] \chi_x(u),$$

where

$$\chi_x(u) = \begin{cases} 1, & u = x; \\ 0, & u \neq x. \end{cases}$$

Obviously, we have

$$(n-r-1) \int_0^{\infty} \left( \int_x^t \left[ f'(x) - \frac{f'(x^+) + f'(x^-)}{2} \right] \chi_x(u) du \right) \sum_{v=0}^{\infty} \omega_{\mu}(v, nx) p_{n-r, v+r}(t) dt = 0.$$

Thus, using above identities, we can write

$$\begin{aligned}
&\left| \frac{(n-2)!}{n^r(n-r-2)!} K_n^{\mu,r}(f, x) - f(x) \right| \\
&\leq \int_0^{\infty} \left| \int_x^t (n-r-1) \sum_{v=0}^{\infty} \omega_{\mu}(v, nx) p_{n-r, v+r}(t) \left( \frac{f'(x^+) + f'(x^-)}{2} + (f')_x(u) \right) du \right| dt \\
&+ \int_0^{\infty} \left| \int_x^t (n-r-1) \sum_{v=0}^{\infty} \omega_{\mu}(v, nx) p_{n-r, v+r}(t) \left( \frac{f'(x^+) - f'(x^-)}{2} \operatorname{sgn}(u - x) \right) du \right| dt. \tag{3.1}
\end{aligned}$$

Also, it can be verified that

$$\begin{aligned} & \left| (n-r-1) \int_0^\infty \left( \int_x^t \frac{f'(x^+) - f'(x^-)}{2} \operatorname{sgn}(u-x) du \right) \sum_{v=0}^\infty \omega_\mu(v, nx) p_{n-r, v+r}(t) dt \right| \\ & \leq \left| \frac{f'(x^+) - f'(x^-)}{2} \right| [T_{n,r,2}^\mu(x)]^{1/2} \end{aligned} \quad (3.2)$$

and

$$\left| (n-r-1) \int_0^\infty \left( \int_x^t \frac{f'(x^+) + f'(x^-)}{2} du \right) \sum_{v=0}^\infty \omega_\mu(v, nx) p_{n-r, v+r}(t) dt \right| = \left| \frac{f'(x^+) + f'(x^-)}{2} \right| T_{n,r,1}^\mu(x). \quad (3.3)$$

Combining (3.1)-(3.3), we have

$$\begin{aligned} \left| \frac{(n-2)!}{n^r(n-r-2)!} K_n^{\mu,r}(f, x) - f(x) \right| & \leq \left| (n-r-1) \int_x^\infty \left( \int_x^t (f')_x(u) du \right) \sum_{v=0}^\infty \omega_\mu(v, nx) p_{n-r, v+r}(t) dt \right| \\ & \quad + \left| (n-r-1) \int_0^x \left( \int_x^t (f')_x(u) du \right) \sum_{v=0}^\infty \omega_\mu(v, nx) p_{n-r, v+r}(t) dt \right| \\ & \quad + \left| \frac{f'(x^+) - f'(x^-)}{2} \right| [T_{n,r,2}^\mu(x)]^{1/2} + \left| \frac{f'(x^+) + f'(x^-)}{2} \right| T_{n,r,1}^\mu(x) \\ & = |A_{n,r}^\mu(f, x)| + |B_{n,r}^\mu(f, x)| \\ & \quad + \left| \frac{f'(x^+) - f'(x^-)}{2} \right| [T_{n,r,2}^\mu(x)]^{1/2} + \left| \frac{f'(x^+) + f'(x^-)}{2} \right| T_{n,r,1}^\mu(x). \end{aligned} \quad (3.4)$$

Applying Remark 1 and Lemma 3, in (3.4), we have

$$\begin{aligned} \left| \frac{(n-2)!}{n^r(n-r-2)!} K_n^{\mu,r}(f, x) - f(x) \right| & \leq |A_{n,r}^\mu(f, x)| + |B_{n,r}^\mu(f, x)| + \left| \frac{f'(x^+) - f'(x^-)}{2} \right| \sqrt{\frac{Cx^2}{n}} \\ & \quad + \left| \frac{f'(x^+) + f'(x^-)}{2} \right| \frac{(1+r)(1-\mu) + x((2+r)(1-\mu) + n\mu)}{(n-r-2)(1-\mu)}. \end{aligned} \quad (3.5)$$

In order to complete the proof of the theorem it sufficient to estimate the terms  $A_{n,r}^\mu(f, x)$  and  $B_{n,r}^\mu(f, x)$ . Applying integration by parts and Lemma 4 with  $y = x - \frac{x}{\sqrt{n}}$ , we have

$$\begin{aligned} |B_{n,r}^\mu(f, r)| & = \left| (n-r-1) \int_0^x \left( \int_x^t (f')_x(u) du d_t(\delta_{n,r}^\mu(x, t)) \right) \right| \\ & = \left| \int_0^x \delta_{n,r}^\mu(x, t) (f')_x(t) dt \right| \\ & \leq \left( \int_0^y + \int_y^x \right) |(f')_x(t)| |\delta_{n,r}^\mu(x, t)| dt \\ & \leq \frac{Cx^2}{n} \int_0^y \bigvee_t^x ((f')_x) \frac{1}{(t-x)^2} dt + \int_y^x \bigvee_t^x ((f')_x) dt \\ & \leq \frac{Cx^2}{n} \int_0^y \bigvee_t^x ((f')_x) \frac{1}{(t-x)^2} dt + \frac{x}{\sqrt{n}} \bigvee_{x-\frac{x}{\sqrt{n}}}^x ((f')_x). \end{aligned}$$

Let  $u = \frac{x}{x-t}$ . Then we have

$$\begin{aligned} \frac{Cx^2}{n} \int_0^y \bigvee_t^x ((f')_x) \frac{1}{(t-x)^2} dt &= \frac{Cx^2}{n} \int_1^{\sqrt{n}} \bigvee_{x-\frac{x}{u}}^x ((f')_x) du \\ &\leq \frac{Cx}{n} \sum_{k=1}^{[\sqrt{n}]} \bigvee_{x-\frac{x}{k}}^x ((f')_x). \end{aligned}$$

Thus

$$|B_{n,r}^\mu(f, r)| \leq \frac{Cx}{n} \sum_{k=1}^{[\sqrt{n}]} \bigvee_{x-\frac{x}{k}}^x ((f')_x) + \frac{x}{\sqrt{n}} \bigvee_{x-\frac{x}{\sqrt{n}}}^x ((f')_x). \quad (3.6)$$

On the other hand, we have

$$\begin{aligned} |A_{n,r}^\mu(f, x)| &= \left| (n-r-1) \int_x^\infty \left( \int_x^t (f')_x(u) du \right) \sum_{v=0}^\infty \omega_\mu(v, nx) p_{n-r, v+r}(t) dt \right| \\ &= \left| (n-r-1) \int_{2x}^\infty \left( \int_x^t (f')_x(u) du \right) \sum_{v=0}^\infty \omega_\mu(v, nx) p_{n-r, v+r}(t) dt \right. \\ &\quad \left. + \int_x^{2x} \left( \int_x^t (f')_x(u) du \right) d_t(1 - \delta_{n,r}^\mu(x, t)) \right| \\ &\leq \left| (n-r-1) \sum_{v=0}^\infty \omega_\mu(v, nx) \int_{2x}^\infty (f(t) - f(x)) p_{n-r, v+r}(t) dt \right| \\ &\quad + |f'(x^+)| \left| (n-r-1) \sum_{v=0}^\infty \omega_\mu(v, nx) \int_{2x}^\infty (t-x) p_{n-r, v+r}(t) dt \right| \\ &\quad + \left| \int_x^{2x} (f')_x(u) du \right| |1 - \delta_{n,r}^\mu(x, 2x)| + \int_x^{2x} |(f')_x(t)| |1 - \delta_{n,r}^\mu(x, t)| dt \\ &\leq (n-r-1) \sum_{v=0}^\infty \omega_\mu(v, nx) \int_{2x}^\infty M t^{2q} p_{n-r, v+r}(t) dt \\ &\quad + \frac{|f(x)|}{x^2} (n-r-1) \sum_{v=0}^\infty \omega_\mu(v, nx) \int_{2x}^\infty p_{n-r, v+r}(t) (t-x)^2 dt \\ &\quad + |f'(x^+)| (n-r-1) \sum_{v=0}^\infty \omega_\mu(v, nx) \int_{2x}^\infty p_{n-r, v+r}(t) |t-x| dt \\ &\quad + \frac{C}{n} |f(2x) - f(x) - x f'(x^+)| \\ &\quad + \frac{Cx}{n} \sum_{k=1}^{[\sqrt{n}]} \bigvee_x^{x+\frac{x}{k}} ((f')_x) + \frac{x}{\sqrt{n}} \bigvee_x^{x+\frac{x}{\sqrt{n}}} ((f')_x). \end{aligned} \quad (3.7)$$

To estimate the integral  $(n-r-1) \sum_{v=0}^{\infty} \omega_{\mu}(v, nx) \int_{2x}^{\infty} Mt^{2q} p_{n-r, v+r}(t) dt$ , in (3.7) above, we proceed as follows: Obviously  $t \geq 2x$  implies that  $t \leq 2(t-x)$  and it follows from Lemma 3, that

$$\begin{aligned} (n-r-1) \sum_{v=0}^{\infty} \omega_{\mu}(v, nx) \int_{2x}^{\infty} Mt^{2q} p_{n-r, v+r}(t) dt &\leq M2^{2q} (n-r-1) \sum_{v=0}^{\infty} \omega_{\mu}(v, nx) \int_0^{\infty} p_{n-r, v+r}(t) (t-x)^{2q} dt \\ &= M2^{2q} T_{n, r, 2q}^{\mu}(x) = O(n^{-q}). (n \rightarrow \infty) \end{aligned}$$

Applying Schwarz inequality and Remark 1, third term in right hand side of (3.7) is estimated as follows:

$$\begin{aligned} &|f'(x^+)| (n-r-1) \sum_{v=0}^{\infty} \omega_{\mu}(v, nx) \int_{2x}^{\infty} p_{n-r, v+r}(t) |t-x| dt \\ &\leq \frac{|f'(x^+)|}{x} (n-r-1) \sum_{v=0}^{\infty} \omega_{\mu}(v, nx) \int_0^{\infty} p_{n-r, v+r}(t) (t-x)^2 dt \\ &= \frac{Cx}{n} |f'(x^+)|. \end{aligned}$$

Thus by Lemma 3 and Remark 1, we have

$$\begin{aligned} |A_{n, r}^{\mu}(f, x)| &\leq O(n^{-q}) + \frac{Cx}{n} |f'(x^+)| + \frac{C}{n} (|f(2x) - f(x) - xf'(x^+)| + |f(x)|) \\ &\quad + \frac{Cx}{n} \sum_{k=1}^{\lfloor \sqrt{n} \rfloor} \bigvee_x^{x + \frac{x}{k}} ((f')_x) + \frac{x}{\sqrt{n}} \bigvee_x^{x + \frac{x}{\sqrt{n}}} ((f')_x). \end{aligned} \quad (3.8)$$

Collecting the estimates (3.5), (3.6) and (3.8), we get the required result.

This completes the proof of Theorem 1.

#### 4. Modification of the operators $K_n^{\mu, r}$ with parameter $c$

In the year 1995, Gupta *et al.* [13] introduced integral modification of the Szász-Mirakyan operators by considering the weight functions of Beta basis functions. Recently, Dubey and Jain [14] modified the operators discussed by Gupta *et al.* [13] with a parameter  $c > 0$  and studied their approximation properties. This type of approach was also discussed by many authors, we refer some papers as [15, 16, 5, 17]. This motivated us to study the rate of convergence for the generalized Jain-Baskakov operators (1.2) with parameter  $c$ , which is defined as the follows:

$$K_{n, c}^{\mu, r}(f, x) = \frac{n^r \Gamma(\frac{n}{c} - r)}{\Gamma(\frac{n}{c} - 1)} \sum_{v=0}^{\infty} \omega_{\mu}(v, nx) \int_0^{\infty} p_{n-rc, v+r}(t, c) f(t) dt, \quad r \geq 0, \quad (4.1)$$

where  $n \in \mathbb{N}$ ,  $n > rc$ , the generalized Baskakov basis function defined as  $p_{n, v}(t, c) = \frac{\Gamma(\frac{n}{c} + v)}{\Gamma(\frac{n}{c}) \Gamma(v + 1)} \frac{(ct)^v}{(1 + ct)^{\frac{n}{c} + v}}$  and  $\omega_{\mu}(v, nx)$  as defined in (1.2). If  $c = 1$ , then the operators (4.1) reduce to the operators defined in (1.2).

**Lemma 5.** Let the  $m^{th}$  order moment be defined as

$$K_{n, r, m}^{\mu, c}(x) = (n - rc - c) \sum_{v=0}^{\infty} \omega_{\mu}(v, nx) \int_0^{\infty} p_{n-rc, v+r}(t) t^m dt. \quad (4.2)$$



Then

$$K_{n,r,0}^{\mu,c}(x) = 1, \quad K_{n,r,1}^{\mu,c}(x) = \frac{nx + (r+1)(1-\mu)}{(n-cr-2c)(1-\mu)}, \text{ for } n > (r+2)c,$$

$$K_{n,r,2}^{\mu,c}(x) = \frac{1}{(n-cr-2c)(n-cr-3c)} \left[ \frac{n^2 x^2}{(1-\mu)^2} + \left[ \frac{n}{(1-\mu)^3} + \frac{n(2r+3)}{1-\mu} \right] x + (r+1)(r+2) \right], \text{ for } n > (r+3)c.$$

The proof of above Lemma follows along the lines of Lemma 2; thus, we omit the details.

**Lemma 6.** Let the  $m^{\text{th}}$  order moment be defined as

$$T_{n,r,m}^{\mu}(x, c) = (n - rc - c) \sum_{v=0}^{\infty} \omega_{\mu}(v, nx) \int_0^{\infty} p_{n-rc, v+r}(t, c) (t - x)^m dt \quad (4.3)$$

$$\text{then } T_{n,r,0}^{\mu}(x, c) = 1, \quad T_{n,r,1}^{\mu}(x, c) = \frac{(1+r)(1-\mu) + x(c(2+r)(1-\mu) + n\mu)}{(n-cr-2c)(1-\mu)}, \text{ for } n > (r+2)c \text{ and}$$

$$\begin{aligned} T_{n,r,2}^{\mu}(x, c) &= x^2 \left( 1 + \frac{n^2}{(n-2c-rc)(n-3c-rc)(1-\mu)^2} - \frac{2n}{(n-2c-rc)(1-\mu)} \right) \\ &+ x \left( \frac{n(1+(3+2r)(1-\mu)^2)}{(n-2c-rc)(n-3c-rc)(1-\mu)^3} - \frac{2(1+r)}{n-2c-rc} \right) \\ &+ \frac{(1+r)(2+r)}{(n-2c-rc)(n-3c-rc)}, \text{ for } n > (r+3)c. \end{aligned}$$

**Remark 3.** For fixed  $r$ , there is a constant  $C_1 > 1$  (which depends only on  $r$ ) such that, for all  $n$  being sufficiently large, all  $\mu$  being sufficiently small (say  $0 < \mu < \mu_0$ ), and  $x \in (0, \infty)$ ,

$$\frac{x^2}{C_1 n} \leq T_{n,r,2}^{\mu}(x, c) \leq \frac{C_1 x^2}{n}.$$

Let  $B_{x^2}[0, \infty) = \{ f : \text{for every } x \in [0, \infty), |f(x)| \leq M_f(1+x^2), M_f \text{ being a constant depending on } f \}$ . By  $C_{x^2}[0, \infty)$ , we denote the subspace of all continuous functions belonging to  $B_{x^2}[0, \infty)$ . Also,  $C_{x^2}^*[0, \infty)$  is subspace of all function  $f \in C_{x^2}[0, \infty)$  for which  $\lim_{x \rightarrow \infty} \frac{f(x)}{1+x^2}$  is finite. The norm on  $C_{x^2}^*[0, \infty)$  is  $\|f\|_{x^2} = \sup_{x \in [0, \infty)} \frac{|f(x)|}{1+x^2}$ .

Examining relation given in Lemma 5 and based on famous Korovkin theorem [18], it is clear that  $\{K_{n,c}^{\mu,r}\}$ ,  $n > rc$  does not form an approximation process. To enjoying of this property, we replace the constant  $\mu$  by a number  $\mu_n \in [0, 1)$  with

$$\lim_{n \rightarrow \infty} \mu_n = 0.$$

Now by Lemma 2.1, we ensure that

$$\lim_{n \rightarrow \infty} K_{n,c}^{\mu_n,r}(t^m, x) = x^m, \quad m = 0, 1, 2$$

uniformly on any interval compact  $E \subset [0, \infty)$ . Base on Korovkin's criteria, we can state the following result:

**Theorem 2.** Let  $K_{n,c}^{\mu_n,r}(f, x)$  with  $n > rc > 0$ , be defined as in (4.1), where  $\lim_{n \rightarrow \infty} \mu_n = 0$ . For any compact set

$E \subset [0, \infty)$  and for each  $f \in C_{x^2}^*[0, \infty)$  one has

$$\lim_{n \rightarrow \infty} K_{n,c}^{\mu_n,r}(f, x) = f(x), \text{ uniformly in } x \in E.$$

**Theorem 3.** Let  $f \in DB_q(0, \infty)$ ,  $q > 0$  and  $x \in (0, \infty)$ . Then for  $C_1 > 1$ , all  $\mu$  being sufficiently small and  $n$  being sufficiently large, we have

$$\begin{aligned} \left| \frac{\Gamma(\frac{n}{c} - 1)}{n^r \Gamma(\frac{n}{c} - r - 1)} (K_{n,c}^{\mu,r}(f, x) - f(x)) \right| &\leq \frac{C_1 x}{n} \left( \sum_{k=1}^{[\sqrt{n}]} \bigvee_{x-x/k}^{x+x/k} ((f')_x) + \frac{x}{\sqrt{n}} \bigvee_{x-x/\sqrt{n}}^{x+x/\sqrt{n}} ((f')_x) \right) \\ &+ \frac{C_1}{n} (|f(2x) - f(x) - x f'(x^+)| + |f(x)|) \\ &+ |f'(x^+)| \frac{C_1 x}{n} + \frac{1}{2} \sqrt{\frac{C_1 x^2}{n}} |f'(x^+) - f'(x^-)| + O(n^{-q}) \\ &+ \frac{1}{2} |f'(x^+) + f'(x^-)| \frac{(1+r)(1-\mu) + x(c(2+r)(1-\mu) + n\mu)}{(n - cr - 2c)(1-\mu)}, \end{aligned}$$

where  $\bigvee_a^b f(x)$  denotes the total variation of  $f_x$  on  $[a, b]$ , and  $f_x$  is defined by

$$f_x(t) = \begin{cases} f(t) - f(x^-), & 0 \leq t < x; \\ 0, & t = x; \\ f(t) - f(x^+), & x < t < \infty. \end{cases}$$

The proof of the above theorem follows along the lines of Theorem 1; thus we omit the details.

**Remark 4.** In [19] Acar et al., estimated the rate of convergence for functions having derivatives of bounded variation in simultaneous approximation of the general integral modification of the Szász-Mirakyan operators having the weight functions of Baskakov basis functions. It would be interesting to extend study of the operators (1.2) in simultaneous approximation. To achieve simultaneous approximation, one has to establish recurrence relation in terms of derivative of the Jain operators  $G_n^\mu(f, x)$ . We are still curious to know, what is recurrence relation of moments of the Jain operators?

**Remark 5.** In year 1983, Stancu [20] generalized the Bernstein polynomials with two parameters  $\alpha$  and  $\beta$  ( $0 \leq \alpha \leq \beta$ ). After this many classical approximation operators generalized into two parameters  $\alpha$  and  $\beta$  [21, 22, 23, 24, 25, 26, 27, 28].

A Stancu generalization of the operators (1.2), is introduced as follows:

$$K_{n,\alpha,\beta}^{\mu,r}(f, x) = \frac{n^r(n-r-1)!}{(n-2)!} \sum_{v=0}^{\infty} \omega_\mu(v, nx) \int_0^\infty p_{n-r,v+r}(t) f\left(\frac{nt+\alpha}{n+\beta}\right) dt, \quad r \geq 0, \quad (4.4)$$

where  $\omega_\mu(v, nx)$  and  $p_{n,v}(t)$  as defined in (1.2). For the operators (4.4), one can study its local approximation properties, Voronovskaja type asymptotic results and rate of convergence for functions having derivatives of bounded

*variation.*

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